APPLIED MATHEMATICS AND STATISTICS LABORATORIES
STANFORD UNIVERSITY
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PRODUCTION PLANNING WITH CONVEX COSTS: A PARAMETRIC STUDY

BY

ARTHUR F. VEINOTT, JR.

TECHNICAL REPORT NO. 66
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PREFACE

Nonmathematical Summary

This study is concerned with the problem of choosing the amounts $x_1, x_2, \ldots, x_n$ of a single product (or aggregation of several products) to produce in each of n successive time periods $1,2,\ldots,n$ so as to minimize the total manufacturing costs over the n periods. The requirements $r_1, r_2, \ldots, r_n$ for the product occurring in periods $1,2,\ldots,n$ are assumed to be known in advance. Requirements in a period are satisfied in so far as possible from stock on hand at the beginning of the period and from production during the period. Requirements which cannot be met in this way are backlogged until they can be satisfied by subsequent production.

Let $y_i = \sum_{j=1}^{i} (x_j - r_j)$. If $y_i \geq 0$, then y_i is the amount of inventory on hand at the end of period i. If y_i is negative, then $-y_i$ is the total amount by which the cumulative requirements exceed the cumulative production in the first i periods. We suppose that this excess, $-y_i$, is backlogged until it can subsequently be satisfied.

Denote the cost of producing x_i units in period i by $c_i(x_i)$. The cost of storing $y_i(\geq 0)$ units at the end of period i is denoted by $h_i(y_i)$. When $y_i < 0$, $h_i(y_i)$ is the penalty cost incurred because $-y_i$ units of requirements are backlogged at the end of period i.

We suppose that there are given upper and lower limits $\bar{x}_i$ and $\underline{x}_i$ $(\underline{x}_i \leq \bar{x}_i)$ respectively on production in period i $(= 1,2,\ldots,n)$. In addition there are given upper and lower limits $\bar{y}_i$ and $\underline{y}_i$ $(\underline{y}_i \leq \bar{y}_i)$

on y_i in period i $(=1,2,\ldots,n-1)$; and no inventory or backlog is allowed at the end of period n, i.e., $y_n = 0$.

The problem is to choose production levels $x_1,\ldots,x_n$ that minimize the total cost

$$\sum_{i=1}^{n} c_i(x_i) + \sum_{i=1}^{n-1} h_i(y_i)$$

over the n periods subject to the above named constraints.

We assume that the cost functions c_i and h_i are convex. For example they might appear as in Figure 1.

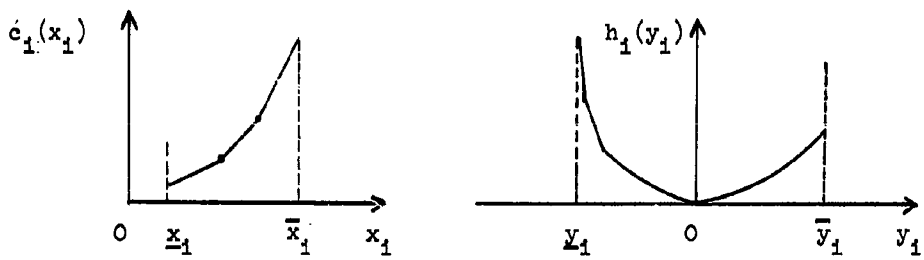

Figure 1

(The functions in Figure 1 are convex because the chord connecting any two points on the graph of either function does not fall below the graph of the function between the two given points.)

The objective of our study is to determine the effect of changes in the requirements and capacity limitations (production, storage, backlog) upon the optimal production levels. We show that the optimal production level in a given period is a non-decreasing function of (1) the requirements in any period, (2) the upper and lower production capacity limits in the given period, and (3) the upper and lower storage limits [1] in the given and all succeeding periods; the optimal production level in a given period is a non-increasing function of (1) the upper and lower production capacity limits in any other period and (2) the upper and lower storage limits in all preceding periods. The above results also lead to simple and efficient computational procedures for finding optimal production quantities.

As an illustration of these methods, suppose that the production levels are assumed to be integers. Suppose also that we have optimal (integer) production levels $x_1,\ldots,x_n$ for the (integer) requirements $r_1,\ldots,r_n$. We seek optimal production levels $x_1',\ldots,x_n'$ for the requirements $r_1',\ldots,r_n'$ where $r_k' \pm 1$ for some k and $r_i' = r_i$ for $i \neq k$. In this circumstance we show that there is an integer j (not necessarily equal to k) for which $x_j' = x_j \pm 1$ and $x_i' = x_i$ for $i \neq j$.

Example:

As an example of the way in which this result can be applied to find an optimal production schedule, suppose the data are as given in table 1 below.

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[1] In this discussion backlogged requirements are viewed as negative inventories.

Table 1

i	r_i	$\bar{x}_i$	$\underline{y}_i$	$\bar{y}_i$	$c_i(1)$	$c_i(2)$	$c_i(3)$	$h_i(-1)$	$h_i(1)$	$h_i(2)$	$h_i(3)$
1	-2	3	-1	2	10	21	33	7	1	3	
2	1	3	0	3	13	26	40		2	5	8
3	3	2	-1	2	9	28		10	1	2	
4	1	0									

Also $c_i(0) = h_i(0) = \underline{x}_i = 0$ for all i.

The idea is to begin by defining a sequence of requirements for which there is only one feasible set of production levels. For example, if $r_1 = -2$, $r_2 = 0$, $r_3 = 1$, $r_4 = 1$ then $x_i = 0$ for $i=1,2,3,4$ is the only feasible set of production levels since $\underline{x}_i = 0$ for all i. We now increase the requirements one unit at a time until we obtain the requirements in table 1. At each stage we find a corresponding optimal collection of production levels.

We begin by increasing the requirements in (say) period 3 to two. From what we have said above, this requires us to produce one additional unit in one of the first four periods. Increasing production in periods one or four is not feasible, the former because the upper inventory limit in period one would be violated and the latter because the upper limit on production in period four would be violated. The cost of producing the one unit in periods two and three is

$$c_2(1) + h_1(2) + h_2(3) + h_3(1) = 13+3+8+1 = 25$$

and

$$c_3(1) + h_1(2) + h_2(2) + h_3(1) = 9+3+5+1 = 18$$

respectively. Thus, the production levels $x_3 = 1$ and $x_i = 0$, $i \neq 3$, are optimal for the requirements $r_1 = -2$, $r_2 = 0$, $r_3 = 2$, $r_4 = 1$. Next we increase the requirements in period two by one. This time the most economical plan is to increase production in period two by one. The final step is to increase requirements in period three again by one unit obtaining $r_1 = -2$, $r_2 = 1$, $r_3 = 3$, $r_4 = 1$. The best plan is now to increase production in period two by one, obtaining $x_1 = 0$, $x_2 = 2$, $x_3 = 1$, $x_4 = 0$ as the final set of optimal production levels.

PRODUCTION PLANNING WITH CONVEX COSTS:
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1. Introduction and Summary

We consider the problem of choosing the amounts $x_1, x_2, \ldots, x_n$ of a single product to produce in each of n successive time periods $1,2,\ldots,n$ so as to minimize the total manufacturing costs over the n periods. The requirements $r_1, r_2, \ldots, r_n$ for the product occurring in periods $1,2,\ldots,n$ are known in advance. Requirements in a period are satisfied in so far as possible from stock on hand at the beginning of the period and from production during the period. Requirements which cannot be met in this way (because, for example, of limited production capacity) are backlogged until they can be satisfied by subsequent production. It is convenient in the sequel to view backlogged requirements as a negative inventory. Similarly, disposal of ("excess") stock is viewed as negative production.

We admit two types of costs in a given period: production and holding, the former being a disposal cost when the production level is negative and the latter being a backlogging cost when the inventory level is negative. These costs are assumed to be convex functions respectively of the quantities produced during and stored at the end of the period. In addition we permit upper and lower limits to be imposed on the amounts produced and stored. The cost functions and quantity limitations for successive periods need not be the same.

The model can be interpreted in a variety of different ways including service scheduling, warehousing decisions, and distribution of effort. Several of these possibilities are discussed in [4] and [7].

Our objective is to study the effect upon the optimal production levels of changes in the parameters, i.e., the requirements and quantity limitations. In the event that the total cost function is strictly convex, the optimal production quantities are unique. For this case our two main results are easily stated as follows. First, the optimal production level in a given period is a non-decreasing function of (1) the requirements in any period, (2) the upper and lower production capacity limits in the given period, and (3) the upper and lower storage limits in the given and all succeeding periods; the optimal production level in a given period is a non-increasing function of (1) the upper and lower production limits in every other period and (2) the upper and lower storage limits in any preceding period. Second, the optimal cumulative production levels $X_1, X_2, \ldots, X_n$ are each non-decreasing functions of the cumulative requirements $R_1, R_2, \ldots, R_n$ in each period, where $X_i = \sum_{j=1}^{i} x_j$ and $R_i = \sum_{j=1}^{i} r_j$ for all i.

The first result is of interest for studies of possible changes in the production or storage capacities, or changes in the minimal guaranteed production level. The first result is also the basis for some efficient parametric programming procedures to be described later. The second result is useful in forecasting. For example suppose that we do not know the actual cumulative requirements $R_1, R_2, \ldots, R_n$, but can forecast maximal and minimal cumulative requirements $\tilde{R}_1, \tilde{R}_2, \ldots, \tilde{R}_n$ and $\underset{\sim}{R}_1, \underset{\sim}{R}_2, \ldots, \underset{\sim}{R}_n$ respectively with assurance that $\underset{\sim}{R}_i \leq R_i \leq \tilde{R}_i$ for all i. We may then compute the corresponding optimal maximal and minimal cumulative production levels $\tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_n$ and $\underset{\sim}{X}_1, \underset{\sim}{X}_2, \ldots, \underset{\sim}{X}_n$ and be assured that the cumulative production levels $X_1, X_2, \ldots, X_n$ that are optimal for $R_1, R_2, \ldots, R_n$ are such that $\underset{\sim}{X}_i \leq X_i \leq \tilde{X}_i$ for all i.

At the beginning of period one we will ordinarily only have to choose X_1 and not subsequent production levels. If it happens that $\underset{\sim}{X}_1 = \tilde{X}_1$ — and this can occur even when the $\tilde{R}_i$ and $\underset{\sim}{R}_i$ are not all the same — then the optimal amount to produce in period one is determined as $\underset{\sim}{X}_1$ without complete knowledge of the R_i. If instead $\underset{\sim}{X}_1 < \tilde{X}_1$, then improved forecasting is needed in order to determine the optimal value of X_1. The value of narrowing the forecast interval in various periods can be assessed by determining its effect upon the width of the interval $[\underset{\sim}{X}_1, \tilde{X}_1]$.

The rather intuitive relations described above between the optimal production levels and the various parameters do not hold in general, but do seem to be valid in a number of situations not encompassed by our hypotheses. It is somewhat surprising, however, that the results may fail to hold when the total costs are convex but cannot be expressed in the form assumed in this paper. For example, suppose the total cost for a two period model is

$$\max(x_2, 3x_1 + 2x_2 - 6) .$$

This function is convex since the maximum of two linear functions is convex. However, if the requirements are $r_1 = r_2 = 1$, then the optimal production levels are $x_1 = 2$, $x_2 = 0$, while if $r_1 = 1$, $r_2 = 3$ the optimal production levels are $x_1 = 1$, $x_2 = 3$ (assuming $X_2 = R_2$ and $x_1 \geq 0$, $x_2 \geq 0$). Observe that neither of our two results holds here.

The fact that the optimal production quantities are non-decreasing functions of the requirements leads immediately to an extremely simple procedure for computing optimal production schedules. In order to justify the algorithm we are about to describe, it is necessary to make two additional assumptions. First, the cost functions are piece-wise linear with the endpoints of each of the intervals on which the functions have linear segments being integers also. Second, the requirements and the upper and lower limits on production and storage are all integers. Under these conditions we show that there are optimal production levels that are also integers. 1/

In the most elemental problem with which our algorithm deals, we start with optimal production levels $x_1, \ldots, x_n$ (all integers) for a given sequence of requirements $r_1, \ldots, r_n$. We then wish to find optimal production quantities $x_1', \ldots, x_n'$ for the new requirements $r_1', \ldots, r_n'$ where $r_i' = r_i + 1$ $(= r_i - 1)$ for some i and $r_j' = r_j$ for $j \neq i$. Our basic result is that there is an integer $k (1 \leq k \leq n)$ for which $x_k' = x_k + 1$ $(= x_k - 1)$ and $x_j' = x_j$ for $j \neq k$. This means that in

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1/ This fact is not surprising since several authors [1], [6] have shown that special cases of our model can be formulated as transportation type linear programming problems for which integer solutions exist. (The amounts "shipped" and "received" must be integers, of course, as they would be under our assumptions.) Our problem can also be formulated in this way.

order to find $\{x_j'\}$, it suffices to compute the total cost under each of the n possible production sequences (corresponding to $k=1,2,\ldots, n$), and then to choose the cheapest one.

Our approach to more complex problems involves solving a sequence of elemental problems. For example, suppose that we seek optimal production levels $x_1',\ldots,x_n'$ for a requirements sequence $r_1',\ldots,r_n'$ for which the r_j' are arbitrary integers. We begin by defining a sequence of requirements $r_1,\ldots,r_n$ for which there is only one feasible set of production levels $x_1,\ldots,x_n$. To illustrate, if disposal of stock is not allowed, we may choose $x_j=r_j=0$ for all j (provided that this choice is feasible). We then construct a new requirements sequence by adding +1(-1) to any r_j for which $(r_j'-r_j)$ is positive (negative). Next we find an optimal production schedule for the new requirements sequence using the technique described in the preceding paragraph. We then repeat the process after replacing the original r_j by the new requirements. We continue in this way until we find that $r_j' = r_j$ for all j, at which point the original problem is solved. The process must terminate because at each stage we reduce $\sum_{j=1}^{n} |r_j'-r_j|$ by +1.

When the r_j' are all non-negative, one can view the above process as satisfying each unit of requirements in turn as cheaply as possible. The special case of this procedure in which requirements must be satisfied in order of their due dates is shown by Johnson [2] to be optimal for the special case of our problem in which no backlogging is allowed, no storage limits are permitted, and the inventory carrying costs are linear.

The parametric programming procedure that we have described above involves only changes in the requirements. In section 4 we develop a generalized procedure in which changes in the production and storage limits are allowed as well.

Alternative procedures for solving the problem considered by Johnson are offered in [4], [5], and [8]. In [9] a generalization of Johnson's procedure is shown to be applicable where the price received for each unit of product sold depends on the quantity sold. 1/ A dynamic programming procedure that solves the general problem in this paper and that neatly exploits the convexity assumptions is given in [4]. References to earlier work will be found in [1], [4], [5], [6], [7], [8], [9].

We outline the plan of this paper. The problem is formulated in section 2. We establish certain fundamental inequalities in section 3. In section 4 we develop computational procedures for finding optimal production levels.

2. Formulation of the Problem

Let y_o be a given constant and let

$$y_i = \sum_{j=1}^{i} (x_j - r_j) + y_o \quad \text{for} \quad i = 1,2,\ldots,n \quad . \tag{1}$$

We interpret $|y_i|$ as the total amount of stock on hand at the end of period i when $y_i \geq 0$, and as the total amount of backlogged requirements at that time when $y_i < 0$. We are implicitly assuming

1/ Actually our model provides for this possibility by allowing disposal of stock. We omit a discussion of this point, however.

that it is not possible to end a period with both a positive inventory and a positive amount of backlogged requirements. Similar remarks apply to production and disposal in a given period.

There are given upper and lower bounds, $\bar{y}_i$ and $\underline{y}_i$ $(\leq \bar{y}_i)$ on y_i, i.e.,

$$\underline{y}_i \leq y_i \leq \bar{y}_i \quad \text{for} \quad i = 1,2,\ldots,n-1 \quad . \tag{2}$$

Observe that by choosing $\underline{y}_i = 0$ for all i that we eliminate all backlogging. We may also allow minimum inventory levels (e.g., as a hedge against uncertainty) by letting $\underline{y}_i$ be positive.

We suppose that

$$\underline{x}_i \leq x_i \leq \bar{x}_i \quad \text{for} \quad i = 1,2,\ldots,n \tag{3}$$

where the $\bar{x}_i$ and $\underline{x}_i$ $(\underline{x}_i \leq \bar{x}_i)$ are given constants. The presence of minimal production levels enables us to study the effect of different guaranteed employment policies.

Let $g(x,r)$ be the total cost associated with the production schedule $x = (x_i)$ and requirements schedule $r = (r_i)$.[1] We assume that g can be written in the form

$$g(x,r) = \sum_{i=1}^{n} c_i(x_i) + \sum_{i=1}^{n-1} h_i(y_i) \tag{4}$$

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[1] We write $z = (z_i)$ to mean that z is a vector whose i^{th} coordinate is z_i.

where c_i and h_i are convex and continuous on the intervals $[\underline{x}_i, \bar{x}_i]$ and $[\underline{y}_i, \bar{y}_i]$ respectively. In applications $c_i(x_i)$ is the cost in period i of producing x_i units if $x_i \geq 0$ and of disposing of $-x_i$ units if $x_i < 0$; $h_i(y_i)$ is the cost in period i of carrying y_i units in inventory if $y_i \geq 0$ and of backlogging $-y_i$ units if $y_i < 0$.

There is no loss of generality in assuming that $y_o = 0$ since, by virtue of (1), (2), and (4), we may replace r_1 by $(r_1 - y_o)$. This substitution could lead to a negative value of r_1 — a possibility that we do not rule out. Indeed we permit any of the r_i to be negative.

Our final assumption is that

$$(5) \qquad y_n = 0 \quad ,$$

which states that we must end period n with no inventory or backlog.

It is convenient to let $\bar{x} = (\bar{x}_i)$, $\underline{x} = (\underline{x}_i)$, $\bar{y} = (\bar{y}_i)$, $\underline{y} = (\underline{y}_i)$, and $p = (-r, \bar{x}, \underline{x}, \bar{y}, \underline{y})$. We call p a parameter set.

In the sequel we shall say that the production schedule x is feasible for the parameter set p (or briefly, x is feasible for p) if the restrictions (1), (2), (3), and (5) are satisfied. If x minimizes (4) subject to the above constraints, we say that x is optimal for p. Finally, we say that p is feasible if there is an x that is feasible for p.

Since the collection of feasible production schedules is compact whenever p is feasible, and since g is continuous, there exists an optimal x whenever p is feasible.

In certain situations it is desirable to drop the assumption (5). When this is the case we shall suppose that there are given limits $\underline{y}_n$ and $\bar{y}_n$ on the inventory level at the end of period n. We can then imbed the original n-period problem in an (n+1)-period problem in which (5) does hold without loss of optimality.

In the extended problem we put $r_{n+1} = \bar{y}_n$, $\bar{x}_{n+1} = \bar{y}_n - \underline{y}_n$, $\underline{x}_{n+1} = 0$, and let $c_{n+1}(z) = 0$ and $h_{n+1}(z) = 0$ for all z. Denote by x any feasible production schedule for the n-period model in which (5) is not imposed. Then in order for the schedule $\tilde{x} = (x, x_{n+1})$ to be feasible for the (n+1)-period model in which (5) does hold, we must have

$$0 = y_{n+1} = y_n + x_{n+1} - r_{n+1}$$

by (1). Hence, $x_{n+1} = r_{n+1} - y_n$. This uniquely defined value of x_{n+1} also satisfies (3) for $i = n+1$ by virtue of the definition of the parameters for period $n+1$.

We now show that x^* is an optimal schedule for the n-period model in which (5) need not hold if and only if $\tilde{x}^* = (x^*, x^*_{n+1})$ is optimal for the (n+1)-period model with (5) holding. First suppose that x^* is optimal for the n-period model and that $\tilde{x} = (x, x_{n+1})$ is any feasible schedule for the (n+1)-period model with requirements schedule $\tilde{r} = (r, r_{n+1})$. Then x is feasible for the n-period model. Also $g(\tilde{x},\tilde{r}) = g(x,r) \geq g(x^*,r) = g(\tilde{x}^*,\tilde{r})$, which was to be shown. Now suppose that $\tilde{x}^*$ is optimal for the (n+1)-period model. Then $g(x,r) = g(\tilde{x},\tilde{r}) \geq g(\tilde{x}^*,\tilde{r}) = g(x^*,r)$, which completes the proof.

The above remarks enable us to confine our discussion in the sequel to situations in which (5) holds.

3. The Fundamental Inequalities

In this section we establish two important inequalities that relate the difference between two parameter sets to the difference between the corresponding optimal production schedules. In order to state the first result it is convenient to introduce a definition. Let $z = (z_i)$ be an $(n-1)$ coordinate vector of real numbers. Let $m(z)$ be the corresponding n coordinate vector of real numbers whose i^{th} $(1 \leq i < n)$ coordinate is $\max_{j \geq i} z_j$ and whose n^{th} coordinate is zero.

Theorem 1

Suppose that $p \equiv (-r, \bar{x}, \underline{x}, \bar{y}, \underline{y})$ and $p' \equiv (-r', \bar{x}', \underline{x}', \bar{y}', \underline{y}')$ are feasible parameter sets and that $p' \leq p$.[1] If $x(x')$ is optimal for $p(p')$, then there is an $x'(x)$ that is optimal for $p'(p)$ and that satisfies

$$(x'-x) \geq -(\bar{x}-\bar{x}')-(\underline{x}-\underline{x}')-m(\bar{y}-\bar{y}')-m(\underline{y}-\underline{y}') \quad . \tag{6}$$

Before proving the theorem we note briefly some of its implications. Observe that if p and p' coincide except that $r' \geq r$, then (6)

1/ Let $v = (v_i)$ and $w = (w_i)$ be vectors of real numbers. We say that $v \geq w$ if $v_i \geq w_i$ for all i.

states that $x' \geq x$. [1] On the other hand if p and p' are identical except that $\bar{x}_i' < \bar{x}_i$ for some i, then (6) states that $x_j' \geq x_j$ for $j \neq i$ and $x_i' \geq x_i-(\bar{x}_i-\bar{x}_i')$. Finally, if p and p' differ only in that $\bar{y}_i' < \bar{y}_i$ for some i, then (6) states that $x_j' \geq x_j-(\bar{y}_i-\bar{y}_i')$ for $j \leq i$ and $x_j' \geq x_j$ for $j > i$.

We remark that the inequality (6) can be sharpened at the expense of complicating its statement. The simpler treatment is adequate for our purposes, however.

Proof:

It is convenient to let $y = (y_i)$ and $z = x-(\bar{x}-\bar{x}')-(\underline{x}-\underline{x}')-m(\bar{y}-\bar{y}')-m(\underline{y}-\underline{y}')$.

We prove only the first part of the theorem, i.e., if x is optimal for p, etc. (The proof of the second part is similar and will not be detailed.) We do so by showing that for any schedule x^* that is feasible for p' and that does not satisfy $x^* \geq z$, there is an alternative schedule x' such that

[1] This special case of theorem 1 was motivated in part by theorem 2', p. 245 in [3], which applies to a generalization of our problem in which the requirements in each period are random variables. Theorem 2' in [3] is not as strong as our theorem 1 for the case of deterministic requirements. In our notation theorem 2' asserts only that $x_1' \geq x_1$. In theorem 2 of this paper we shall show that the inequality $x_1' \geq x_1$ holds under the weaker condition that $R_i' \geq R_i$ for all i.

(i) x' is feasible for p';

(ii) $g(x',r') \leq g(x^*,r')$;

(iii) $(x'-x,x'-z,y'-y)$ has one (or more) fewer non-zero coordinates than (x^*-x,x^*-z,y^*-y). [1]

Assuming for the moment the truth of the above result, observe that either $x' \geq z$ or, upon replacing x^* by x', that it is possible to construct a new x' having the properties (i), (ii), (iii). Since at each repetition of this procedure we either terminate by finding an x' satisfying (i) and (ii) and having the property that $x' \geq z$, or we reduce the number of non-zero coordinates of $(x'-x, x'-z,y'-y)$ by one, we must terminate in at most $3n-1$ steps. To complete the proof of the first assertion of the theorem, it is sufficient to observe that if x^* is optimal for p', the x' that is obtained at the termination of the above process satisfies the conclusions of the theorem.

It remains to describe a procedure for constructing a schedule x' with the properties (i), (ii), (iii). Since $x^* \not\geq z$, there exists an i, $1 \leq i \leq n$, for which $x_i^* < z_i$. It is convenient now to consider two cases, $y_{i-1}^* \leq y_{i-1}$ and $y_{i-1}^* > y_{i-1}$.

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[1] In order to conserve space we have not stated explicitly that y^* is the vector of inventories associated with x^* and r' because the notation makes this fact evident. For the same reason we shall subsequently refer to y_i^* without identifying it as the i^{th} coordinate of y^*. We shall follow a similar practice hereafter when no ambiguity will result.

Case 1: $y^*_{i-1} \leq y_{i-1}$

Let j be the smallest integer greater than i for which $x^*_j > x_j$. Such an integer must exist for if not

$$y^*_n = y^*_{i-1} + \sum_{t=i}^{n} (x^*_t - r'_t) < y_{i-1} + \sum_{t=i}^{n} (x_t - r_t) = y_n = 0$$

which contradicts (5). The inequality follows from the fact that $y^*_{i-1} \leq y_{i-1}$; $x^*_i < z_i \leq x_i$; $x^*_t \leq x_t$ for $t = i+1, i+2, \ldots, n$; and $r'_t \geq r_t$ for all i.

Let $\epsilon = \min(z_i - x^*_i,\ x^*_j - x_j)$. It follows from the preceding discussion that $\epsilon > 0$. In addition, as we now show,

$$(7) \qquad x_i - x^*_i \geq \epsilon;\ y_t - y^*_t \geq \epsilon \ \text{ for } \ t=i,i+1,\ldots,j-1; \ \text{ and } \ x^*_j - x_j \geq \epsilon .$$

The first and last inequalities follow from the definition of ϵ. The intermediate inequalities follow from the first inequality and

$$(8) \qquad y_t - y^*_t = y_{i-1} - y^*_{i-1} - \sum_{k=i}^{t} (r_k - r'_k) + \sum_{k=i}^{t} (x_k - x^*_k) \geq x_i - x^*_i \quad .$$

Now define an alternative production schedule x' by $x' = x^* + \epsilon(u_i - u_j)$, where u_k is the k^{th} coordinate unit vector, i.e., the k^{th} coordinate of u_k is $+1$ and all other coordinates are zero. We show that x' is feasible for p' as follows. Using (6), (8), the definition of ϵ, the hypotheses of the theorem, and the feasibility of x and x^* respectively for p and p', we have

(9) $$\underline{x}_i' \le x_i^* < x_i^* + \epsilon = x_i' \le z_i \le x_i - (\bar{x}_i - \bar{x}_i') \le \bar{x}_i'$$

and

$$\underline{x}_j' \le \underline{x}_j \le x_j \le x_j^* - \epsilon = x_j' < x_j^* \le \bar{x}_j' \ .$$

Thus since $x_t' = x_t^*$ for $t \ne i,j$, x' satisfies (3). Also for $i \le t < j$,

$$\begin{aligned} \underline{y}_t' \le \underline{y}_t^* < y_t^* + \epsilon = y_t' &\le y_t + x_i^* - x_i + \epsilon \\ &\le y_t + x_i^* - z_i - (\bar{y}_t - \bar{y}_t') + \epsilon \\ &\le x_i^* - z_i + \bar{y}_t' + \epsilon \le \bar{y}_t' \ . \end{aligned}$$

For all other t, $y_t' = y_t^*$, so that x' satisfies (2) and (5). Thus x' satisfies (1).

In the following we denote by $D^+f(x)$ and $D^-f(x)$ respectively the right and left hand derivatives of a function f at the point x. When the function is convex, as will be the case in the succeeding discussion, we may be assured that the right and left hand derivatives exist.

Using the convexity of c_t and h_t and (7) we have

$$\begin{aligned} g(x^*,r') - g(x',r') &= c_i(x_i^*) - c_i(x_i^*+\epsilon) \\ &+ \sum_{k=i}^{j-1} [h_k(y_k^*)-h_k(y_k^*+\epsilon)] + c_j(x_j^*)-c_j(x_j^*-\epsilon) \\ &\ge \epsilon[-D^-c_i(x_i^*+\epsilon) - \sum_{k=i}^{j-1} D^-h_k(y_k^*+\epsilon) + D^+c_j(x_j^*-\epsilon)] \\ &\ge \epsilon[-D^-c_i(x_i) - \sum_{k=i}^{j-1} D^-h_k(y_k) + D^+c_j(x_j)] \ . \end{aligned}$$

In view of this inequality we may establish property (ii) by showing that

$$(10) \qquad -D^{-}c_i(x_i) - \sum_{k=i}^{j-1} D^{-}h_k(y_k) + D^{+}c_j(x_j) \geq 0 \quad .$$

To prove (10) we show first that the schedule $x(\delta) = x-\delta(u_i - u_j)$ is feasible for p provided that $0 \leq \delta \leq \epsilon$. Assuming for the moment that we have done this, we note that x is optimal for p and therefore that

$$\frac{g(x(\delta),r)-g(x,r)}{\delta} \geq 0 \quad \text{for} \quad 0 < \delta \leq \epsilon \ .$$

Letting $\delta \to 0+$, the above inequality becomes (10).

It remains to establish the feasibility of $x(\delta)$. Using (6), (8), (9), the definition of ϵ, the hypotheses of the theorem, and the feasibility of x and x^* respectively for p and p', we have

$$\underline{x}_i \leq x_i - z_i + \underline{x}_i' \leq x_i-\epsilon \leq x_i(\delta) \leq x_i \leq \bar{x}_i$$

and

$$\underline{x}_j \leq x_j \leq x_j(\delta) \leq x_j + \epsilon \leq x_j^* \leq \bar{x}_j' \leq \bar{x}_j \ .$$

Also for $i \leq t < j$,

$$\underline{y}_t \leq \underline{y}_t' - z_i + x_i \leq \underline{y}_t' - z_i + y_t - y_t^* + x_i^*$$

$$\leq y_t - z_i + x_i^* \leq y_t - \epsilon \leq y_t(\delta) \leq y_t \leq \bar{y}_t \ .$$

For $t \neq i,j$, $x_t(\delta) = x_t$ and for $t < i$ and $t \geq j$, $y_t(\delta) = y_t$, so that $x(\delta)$ is feasible for p as claimed.

To establish (iii) we observe from the definition of ϵ that all but the i^{th} and j^{th} coordinates of $(x'-x)$ and (x^*-x), and of $(x'-z)$ and (x^*-z) are identical. Also all but the i^{th} through the $(j-1)^{th}$ coordinates of $(y'-y)$ and (y^*-y) are the same. On the other hand the i^{th} and j^{th} coordinates of (x^*-x) and (x^*-z), and the i^{th} through the $(j-1)^{th}$ coordinates of (y^*-y) are all non-zero. But at least one of these coordinates in $(x'-x)$, $(x'-z)$, or $(y'-y)$ is zero, which proves (iii).

<u>Case 2</u>: $\qquad y^*_{i-1} > y_{i-1}$

Let j be the largest integer less than i for which $x^*_j > x_j$. Such an integer must exist for otherwise

$$y^*_{i-1} = \sum_{t=1}^{i-1} (x^*_t - r'_t) \leq \sum_{t=1}^{i-1} (x_t - r_t) = y_{i-1}$$

if $i \geq 2$ and $y^*_0 = y_0 = 0$ if $i = 1$, which is a contradiction.

Let $\epsilon = \min(z_i - x^*_i,\ x^*_j - x_j,\ y^*_{i-1} - y_{i-1})$. Clearly $\epsilon > 0$. In addition

(11) $\qquad x^*_j - x_j \geq \epsilon \ ; \ y^*_t - y_t \geq \epsilon \ \text{ for } \ j \leq t \leq i-1 \ ; \ x_i - x^*_i \geq \epsilon$.

The first and last inequalities follow from the definition of ϵ. It remains only to observe that for $j \leq t \leq i-1$,

$$y_t^*-y_t = y_{i-1}^*-y_{i-1} - \sum_{k=t+1}^{i-1} (x_k^*-x_k) + \sum_{k=t+1}^{i-1} (r_k'-r_k) \geq y_{i-1}^*-y_{i-1} \geq \epsilon$$

which establishes (11).

We define an alternative schedule x' for p' by $x' = x^*+\epsilon(u_i-u_j)$. The proof that x' satisfies (3) is precisely the same as for case 1. Now for $j \leq t < i$, we have using (11), the hypotheses of the theorem, and the feasibility of x and x^* respectively for p and p' that

$$\underline{y}_t' \leq \underline{y}_t \leq y_t \leq y_t^*-\epsilon = y_t' < y_t^* \leq \bar{y}_t' \quad ,$$

while for all other t, $y_t^* = y_t'$, so that x' satisfies (2) and (5). Thus property (i) is established.

If we now employ (11) and the convexity of the c_t and h_t, we find that

$$\begin{aligned} g(x^*,r') - g(x',r') &= c_j(x_j^*) - c_j(x_j^*-\epsilon) \\ &+ \sum_{k=j}^{i-1} [h_k(y_k^*) - h_k(y_k^*-\epsilon)] + c_i(x_i^*) - c_i(x_i^*+\epsilon) \\ &\geq \epsilon[D^+c_j(x_j^*-\epsilon) + \sum_{k=j}^{i-1} D^+h_k(y_k^*-\epsilon) - D^-c_i(x_i^*+\epsilon)] \\ &\geq \epsilon[D^+c_j(x_j) + \sum_{k=j}^{i-1} D^+h_k(y_k) - D^-c_i(x_i)] \quad . \end{aligned}$$

In order to verify property (ii) it therefore suffices to show that

$$D^+c_j(x_j) + \sum_{k=j}^{i-1} D^+h_k(y_k) - D^-c_i(x_i) \geq 0 \; . \tag{12}$$

Let $x(\delta) = x-\delta(u_i-u_j)$. We show that $x(\delta)$ is feasible for p provided that $0 \leq \delta \leq \epsilon$. The fact that $x(\delta)$ satisfies (3) follows from the same argument used in case 1. For $j \leq t < i$, we have from (11), the hypotheses of the theorem, and the feasibility of x and x^* respectively for p and p' that

$$\underline{y}_t \leq y_t \leq y_t(\delta) \leq y_t^* \leq \bar{y}_t' \leq \bar{y}_t \quad ,$$

and $y_t(\delta) = y_t$ for all other t. Thus, $x(\delta)$ is feasible for p.

Since x is optimal for p,

$$\frac{g(x(\delta),r)-g(x,r)}{\delta} \geq 0 \quad \text{for} \quad 0 < \delta \leq \epsilon \ .$$

Letting $\delta \to 0+$, we obtain (12).

The proof that x' satisfies (iii) is exactly the same as for case 1 upon interchanging the roles of i and j. This completes the proof.

In order to state and prove our next theorem it will be convenient to introduce notation for the cumulative production and requirements schedules associated respectively with the production and requirements schedules x and r. In particular let

$$X = (X_1, X_2, \ldots, X_n) \quad \text{and} \quad R = (R_1, R_2, \ldots, R_n) \ .$$

Theorem 2

Suppose that $p \equiv (-r, \bar{x}, \underline{x}, \bar{y}, \underline{y})$ and $p' \equiv (-r', \bar{x}, \underline{x}, \bar{y}, \underline{y})$ are feasible parameter sets and that $R' \geq R$. If $x(x')$ is optimal for $p(p')$, then there is an $x'(x)$ that is optimal for $p'(p)$ and that satisfies $X' \geq X$.

Proof:

The proof of the first assertion in the theorem (i.e., if x is optimal for p, etc.) consists in showing that for any schedule x^* that is feasible for p' and for which $X^* \not\geq X$, there is an alternative schedule x' that has the properties (i), (ii), (iii) (let $z = x$ in (iii)) given in the proof of theorem 1. Once this construction is justified, the first assertion of the theorem follows from an obvious adaptation of the corresponding part of the proof of theorem 1. It remains, therefore, to develop a procedure for constructing a schedule x' with the properties (i), (ii), (iii).

Since $X^* \not\geq X$, there is an integer i for which $X_i^* < X_i$. Let i be the smallest such integer. Denote by j the smallest integer greater than i for which $X_j^* \geq X_j$. The integer j exists because by (5)

$$X_n^* - X_n = R_n' - R_n \geq 0 .$$

Let $\epsilon = \min[x_i - x_i^*,\ x_j^* - x_j,\ \min_{i \leq k < j} (y_k - y_k^*)]$. We have $\epsilon > 0$ since

$$x_i - x_i^* = (X_i - X_{i-1}) - (X_i^* - X_{i-1}^*) > 0$$

$$x_j^* - x_j = (X_j^* - X_{j-1}^*) - (X_j - X_{j-1}) > 0 \quad \text{and}$$

$$y_k - y_k^* = (X_k - R_k) - (X_k^* - R_k') > 0 \qquad \text{for } i \leq k < j .$$

Now define the alternative schedule x' for p' by $x' = x^* + \epsilon(u_i - u_j)$. Thus $y_k' = y_k^* + \epsilon$ for $k = i, i+1, \ldots, j-1$ and $y_k' = y_k^*$ otherwise. Using these calculations it follows easily that x' satisfies (i).

It is now possible to show that (ii) and (iii) hold using precisely the same argument as that used to establish the same properties in case 1 of the proof of theorem 1. This completes the proof of the first assertion of the theorem. The proof of the second assertion of the theorem follows in a similar manner.

4. A Parametric Programming Procedure

Theorem 1 plays a central role in this section in the development of procedures for finding optimal production schedules. Our techniques are especially useful when we seek optimal production schedules for several different parameter sets. This is because we use the information gained in determining an optimal production schedule for one parameter set in an efficient way to reduce the computations needed to find an optimal production schedule for a different parameter set.

In order to simplify the exposition in this section, it is convenient to impose the following assumption:

P: $c_i(x_i)$ and $h_i(y_i)$ are each piecewise-linear functions with the endpoints of each of the intervals on which the functions have linear segments being integers also.

The basic result on which the computational procedures of this section rest is

Theorem 3

Suppose that $p \equiv (-r, \overline{x}, \underline{x}, \overline{y}, \underline{y})$ and $p' \equiv (-r', \overline{x}', \underline{x}', \overline{y}', \underline{y}')$ are feasible parameter sets with integral coordinates, that $(p-p')$ is a unit vector, that P holds, and that $x(x')$ has integral coordinates and is optimal for $p(p')$.

(a) If $(\bar{y}', \underline{y}') = (\bar{y}, \underline{y})$, then there is an $x'(x)$ that is optimal for $p'(p)$ and that has the property that $(x'-x) + (\bar{x}-\bar{x}') + (\underline{x}-\underline{x}')$ is a unit vector.

(b) If $(\bar{y}-\bar{y}') + (\underline{y}-\underline{y}') = u_k$ for some k, then there is an $x'(x)$ that is optimal for $p'(p)$ and for which either $(x'-x) = u_j - u_i$ for some $i \leq k < j$, or $(x'-x) = 0$. [1/]

We defer the proof of the theorem briefly in order to explore some of its implications. The value of part (a) is that when we are given an x that is optimal for p, we can find an x' that is optimal for p' (where the upper and lower bounds on inventory levels are unchanged) by considering only the n production schedules obtained by separately adding each of the n unit vectors to $x-(\bar{x}-\bar{x}') - (\underline{x}-\underline{x}')$. On the other hand if we are told that x' is optimal for p', part (a) assures us that we need compare only the n production schedules obtained by separately subtracting each of the n unit vectors from $x' + (\bar{x}-\bar{x}') + (\underline{x}-\underline{x}')$

<u>Example 1</u>:

If $x = (3,0,1)$ is optimal for p where $r = (-2,6,0)$, $\bar{x} = (3,3,3)$, $\underline{x} = 0$, $\bar{y} = (5,7)$, and $\underline{y} = (0,-2)$, and if p' is such that $r' = (-1,6,0)$ and $(\bar{x}', \underline{x}', \bar{y}', \underline{y}') = (\bar{x}, \underline{x}, \bar{y}, \underline{y},)$, then one of the three production

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1/ In this part u_k has n-1 coordinates while u_i and u_j have n coordinates.

schedules $(4,0,1)$, $(3,1,1)$, $(3,0,2)$ is optimal for p'. In order to find which one is optimal we first eliminate the infeasible schedule(s) – in this case $(4,0,1)$ – and then compute the cost associated with each of the remaining schedules.

Example 2:

Suppose in example 1 that we let p be defined as before and let $\underline{x}' = (0,1,0)$ and $(-r', \overline{x}', \overline{y}', \underline{y}') = (-r, \overline{x}, \overline{y}, \underline{y})$. (Observe that these definitions require us to interchange the roles of p and p' since $(p'-p)$ is a unit vector.) Then one of the three production schedules $(2,1,1)$, $(3,0,1)$ $(3,1,0)$ is optimal for p'. Of these schedules, only the second is infeasible.

Part (b) of the theorem states in part that if we have at hand an x that is optimal for p, if one of the upper or lower limits on inventory is reduced, and if none of the other parameters is changed, then we can find an x' that is optimal for p' by comparing the $k(n-k)$ schedules $x + u_j - u_i$, $i \leq k < j$, and the schedule x. This comparison is almost as easy to perform as that in part (a), even though superficially the number of comparisons may seem to be as high as $\frac{n^2}{4} - 1$ (when $k = n/2$) rather than $n-1$ as in part (a). The reason for this is that the optimal values of i and j may be chosen independently. This is because whatever the choice of i and j, the inventory on hand at the end of period k is $y_k - 1$ when $x + u_j - u_i$ is the production schedule. The first k periods and the last $n-k$ periods can therefore be thought of as two separate sub-problems for the purposes of this computation. Similar remarks apply when we are given an x' that is optimal for p'.

To summarize, theorem 3 provides a simple procedure for finding an optimal x' for p' once we are given an optimal x for p provided that $\pm(p-p')$ is a unit vector. The procedure generalizes easily to situations in which $\pm(p-p')$ is not a unit vector but where the other hypotheses of the theorem are retained. The technique is to successively add and subtract unit vectors from p until p' is obtained. Optimal schedules are found in order for each intermediate parameter set using theorem 3. Formally, denote by $p^1, p^2, \ldots, p^{m-1}$ the sequence of intermediate parameter sets produced in the process of successively modifying p by adding and subtracting unit vectors. Upon letting $p^0 = p$ and $p^m = p'$, we see that each $p^i (1 \leq i \leq m)$ is obtained by adding to or subtracting from p^{i-1} a suitable unit vector. 1/

If we have at hand an x^0 that has integral coordinates and that is optimal for $p^0(=p)$, we can determine in order of sequence $x^1, x^2, \ldots, x^m$ of optimal production schedules for $p^1, p^2, \ldots, p^m$ respectively. Each $x^i (1 \leq i \leq m)$ is formed from x^{i-1} by applying the appropriate part of theorem 3.

1/ The choice of the p^i is essentially arbitrary. Thus, if optimal production schedules are needed for several parameter sets, it may be desirable to define the sequence $\{p^i\}$ so as to include these sets. This may necessitate adding and subtracting the same unit vector in the course of the computations. This creates no difficulties but does increase the number of intermediate parameter sets and hence the amount of computation. For a sequence $\{p^i\}$ in which no unit vector is both added and subtracted, the number of intermediate parameter sets will be

$$m-1 = \sum_{i=1}^{n} |p_i'-p_i| - 1 \quad .$$

Two questions remain unanswered by the above discussion. First, how can we find a p^o for which an optimal x^o is easily determined? Second, what can be done if we find an intermediate parameter set p^i, say, that is infeasible?

We defer discussion of the second question until after we have proved theorem 3. One answer to the first question is to define p^o so that only one feasible production schedule exists. That schedule is necessarily optimal and provides a starting point for the computations. Two illustrations of this idea are given below. Let

$$\bar{X}_i = \sum_{j=1}^{i} \bar{x}_j \quad \text{and} \quad \underline{X}_i = \sum_{j=1}^{i} \underline{x}_j \quad .$$

<u>Start 1</u>:

Suppose that $x^o \geq 0$ is an arbitrary production schedule with integral coordinates for which $X_n^o = R_n'$. In many cases it will be natural to let x^o be a "good guess" at an optimal solution. If we choose $\underline{x}^o = x^o = \bar{x}^o$, $r^o = r'$, $\bar{y}_i^o = \max(\bar{y}_i', y_i^o)$ and $\underline{y}_i^o = \min(\underline{y}_i', y_i^o)$ for all i, then x^o is the only feasible production schedule for $p^o = (-r^o, \bar{x}^o, \underline{x}^o, \bar{y}^o, \underline{y}^o)$.

<u>Start 2</u>:

Suppose that $x^o = \underline{x}'(\bar{x}')$ and that r^o is such that $r^o \leq r'(r^o \geq r')$ and $R_n^o = \underline{X}_n'(\bar{X}_n')$. Let $\bar{x}^o = \bar{x}'(\underline{x}^o = \underline{x}')$, $\bar{y}_i^o = \max(\bar{y}_i', y_i^o)$ and $\underline{y}_i^o = \min(\underline{y}_i', y_i^o)$ for all i. Again x^o is feasible and no other schedules have this property.

The above starting points are intended to be suggestive and naturally do not exhaust all the possibilities. For example, an obvious counterpart of start 1 is to set $\underline{y}^o = y^o = \bar{y}^o$.

We now give a

Proof of theorem 3-part a:

We consider only the case where an optimal x is given for p. The proof of the other case is similar and is therefore omitted.

By the first assertion of theorem 1 — and this is the key point — there is an x' that is optimal for p' and that satisfies $x' \geq z$, where $z = x-(\bar{x}-\bar{x}')-(\underline{x}-\underline{x}')$. The remainder of the proof consists in showing that $g(x',r')$ is linear in x' for all feasible x' for which $x' \geq z$. The theorem follows easily from this fact.

It is convenient to let $Z_i = \sum_{j=1}^{i} z_j$ for all i. We show that

$$R'_n = Z_n + 1 \, . \tag{13}$$

There are two cases, $r' > r$ and $r' = r$. In the former event

$$R'_n = R_n + 1 = X_n + 1 = Z_n + 1 \, .$$

In the latter case

$$R'_n = R_n = X_n = Z_n + 1 \, .$$

In view of (2), (3), (13), and the fact that $x' \geq z$, we have

$$\max(\underline{x}'_i, z_i) \leq x'_i \leq \min(\bar{x}'_i, z_i+1) \quad \text{for } i=1,2,\ldots,n \tag{14}$$

and

$$(15) \qquad \max(\underline{y}_i, Z_i - R_i') \leq X_i' - R_i' \leq \min(\bar{y}_i, Z_i - R_i' + 1) \quad \text{for} \quad i=1,2,\ldots,n-1 \ .$$

Since p, p', and x have integral coordinates, the upper and lower limits in (14) are integers, and they differ at most by one. A similar remark is applicable to (15). Hence by P, there are numbers $a_o, a_1, \ldots, a_n$ such that

$$g(x',r') = a_o + \sum_{i=1}^{n} a_i \, x_i'$$

for all feasible $x' \geq z$. Denote by S the set of all indices i for which there is a feasible $x' \geq z$ with $x_i' > z_i$. For these i, $x' = z + u_i$ is also feasible. The set S is not empty, because otherwise no feasible x' would exist by theorem 1, contradicting an hypothesis of theorem 3. Now let i be an integer in S for which a_i is a minimum. Clearly $x' = z + u_i$ is then optimal, which proves part (a).

Before proving part (b) we digress to establish a lemma that will enable us to sharpen the results of theorem 1. Consider the problem of finding a vector x^* that minimizes a convex function $f(x)$ subject to the constraints

$$(16) \qquad a_i x \leq b_i, \quad i=1,2,\ldots,n \ .$$

where the a_i and x are vectors and the b_i are scalars. Let S be the set of indices i for which $a_i x^* < b_i$. Also consider the same problem where we replace the b_i by b_i', i.e., (16) becomes

$$(17) \qquad a_i x \leq b_i' \ , \quad i=1,2,\ldots,n \ .$$

Lemma 1

(a) If $b_i' = b_i$ for $i \notin S$ and $b_i' \geq a_i x^*$, for $i \in S$, then x^* minimizes $f(x)$ subject to (17).

(b) If $b_j' < b_j$ for some $j \notin S$, and if $b_i' = b_i$ otherwise, then there is an x' that minimizes $f(x)$ subject to (17) for which $a_j x' = b_j'$ (provided that (17) is feasible).

(c) If $b_j' > b_j$ for some $j \notin S$ and if $b_i' = b_i$ otherwise, then there is an x' that minimizes $f(x)$ subject to (17) for which $b_j \leq a_j x' \leq b_j'$.

Proof:

We begin with part (a). Suppose the contrary, that is x' satisfies (17) and $f(x') < f(x^*)$. Since x^* satisfies (17), so does $x'' = \alpha x' + (1-\alpha)x^*$ for $0 \leq \alpha \leq 1$. Denote by B the set of indices i for which $a_i x' > b_i$. Clearly B is not empty for otherwise x^* could not minimize $f(x)$ subject to (16). Let

$$\alpha = \min_{i \in B} \frac{b_i - a_i x^*}{a_i(x' - x^*)} .$$

Observe that $0 < \alpha < 1$. Also for $i \notin B$,

$$a_i x'' = \alpha\, a_i x' + (1-\alpha) a_i x^* \leq \alpha\, b_i + (1-\alpha) b_i = b_i$$

and for $i \in B$

$$a_i x'' = \alpha\, a_i x' + (1-\alpha) a_i x^* = \alpha\, a_i (x' - x^*) + a_i x^*$$
$$\leq (b_i - a_i x^*) + (a_i x^*) = b_i .$$

Therefore x'' satisfies (16) and

$$f(x'') \leq \alpha f(x') + (1-\alpha)f(x^*) < \alpha f(x^*)+(1-\alpha)f(x^*) = f(x^*) ,$$

which contradicts the optimality of x^*.

We also prove part (b) by contradiction. Thus suppose that x' minimizes $f(x)$ subject to (17), that $a_j x' < b_j'$, and that $f(x') < f(x)$ for all x satisfying (17) for which $a_j x = b_j'$. Let

$$x'' = \alpha x' + (1-\alpha)x^* \quad \text{where} \quad \alpha = \frac{b_j - b_j'}{b_j - a_j x'} .$$

Clearly $a_j x'' = b_j'$ and $a_i x'' \leq b_i'$ for $i \neq j$. Also $f(x') \geq f(x^*)$. Therefore

$$f(x'') \leq \alpha f(x')+(1-\alpha)f(x^*) \leq f(x')$$

which is a contradiction. This completes the proof. The proof of part (c) is similar and is omitted.

As an example of the way in which part (a) of the lemma can be applied to our problem, suppose that the production schedule x is optimal for p and that, say $\underline{y}_i < y_i$. Then x is also optimal for all $\underline{y}_i$ for which $\underline{y}_i \leq y_i$. Analogous remarks apply to variations of $\bar{y}_i$, $\bar{x}_i$, and $\underline{x}_i$. The usefulness of parts (b) and (c) of the lemma will become apparent in our proof of part (b) of theorem 3.

<u>Proof of theorem 3-part b</u>:

As usual, we consider only the case where an optimal x is given for p.

As a preliminary we show that there is an x' that is optimal for p' and that satisfies

(18) $$y_k' \leq y_k$$

and

(19) $$x_i' \leq x_i \quad \text{for} \quad i=1,2,\ldots,k \ .$$

The property (18) follows from parts (a) and (b) of lemma 1. As a consequence if $\bar{y}_k' > y_k$, we may instead let $\bar{y}_k' = y_k$ without loss of optimality.

The proof that (19) holds for the revised parameter set consists in showing that for any x^* that is feasible for p' (revised) and for which (19) does not hold, there is an alternative schedule x' that has the properties (i), (ii), (iii) (let $z = x$ in (ii)) given in the proof of theorem 1. Once this construction is justified, (19) follows from an obvious adaptation of the corresponding part of the proof of theorem 1.

Suppose x^* does not satisfy (19). Then there is an integer i, $1 \leq i \leq k$, for which $x_i^* > x_i$. We consider two cases, $y_{i-1}^* < y_{i-1}$ and $y_{i-1}^* \geq y_{i-1}$.

<u>Case 1</u>: $y_{i-1}^* < y_{i-1}$.

Let j be the largest integer smaller than i for which $x_j^* < x_j$. Such an integer must exist for if not

$$y_{i-1}^* = \sum_{t=1}^{i-1} (x_t^*-r_t) \geq \sum_{t=1}^{i-1} (x_t-r_t) = y_{i-1} \quad ,$$

which is a contradiction.

Let $\epsilon = \min(x_i^*-x_i,\ y_{i-1}-y_{i-1}^*,\ x_j-x_j^*)$. Clearly $\epsilon > 0$ and

(20) $\quad x_i^*-x_i \geq \epsilon;\quad y_t-y_t^* \geq \epsilon \quad$ for $\quad t = j,j+1,\ldots,i-1 \quad$ and $\quad x_j-x_j^* \geq \epsilon$.

Consider the alternative production schedule x' defined by $x' = x^* + \epsilon(u_j-u_i)$. It is easy to show that x' is feasible for p' by using (20) and the feasibility of x and x^* respectively for p and p'.

Now in the usual way

$$
\begin{aligned}
g(x^*,r)-g(x',r) &= c_j(x_j^*)-c_j(x_j^*+\epsilon) \\
&+ \sum_{t=j}^{i-1} [h_t(y_t^*)-h_t(y_t^*+\epsilon)] + c_i(x_i^*)-c_i(x_i^*-\epsilon) \\
&\geq \epsilon[-D^-c_j(x_j) - \sum_{t=j}^{i-1} D^-h_t(y_t) + D^+c_i(x_i)] \geq 0 \quad .
\end{aligned}
$$

The final inequality follows from the fact that for $0 \leq \delta \leq \epsilon$, $x(\delta) = x-\delta(u_j-u_i)$ is feasible for p and $g(x(\delta),r) \geq g(x,r)$.

We have now established properties (i) and (ii). Property (iii) follows using the same argument given in proving theorem 1.

__Case 2:__ $\qquad y_{i-1}^* \geq y_{i-1}$

Let j be the smallest integer greater than i and not exceeding k for which $x_j^* < x_j$. The integer j exists because if not,

$$y_k^* = y_{i-1}^* + \sum_{t=i}^{k} (x_t^*-r_t) > y_{i-1} + \sum_{t=i}^{k} (x_t-r_t) = y_k \ ,$$

which contradicts (18).

Let $\epsilon = \min(x_i^*-x_i,\ x_j-x_j^*)$. Clearly $\epsilon > 0$ and

(21) $\quad x_i^*-x_i \geq \epsilon$; $y_t^*-y_t \geq \epsilon$ for $t=i,i+1,\ldots,j-1$; and $x_j-x_j^* \geq \epsilon$.

The alternative schedule $x' = x^*+\epsilon(u_j-u_i)$ is feasible for p' as can be shown using (21). Also

$$g(x^*,r)-g(x',r) = c_i(x_i^*)-c_i(x_i^*-\epsilon)$$

$$+\sum_{t=i}^{j-1} [h_t(y_t^*)-h_t(y_t^*-\epsilon)] + c_j(x_j^*)-c_j(x_j^*+\epsilon)$$

$$\geq \epsilon[D^+c_i(x_i) + \sum_{t=i}^{j-1} D^+h_t(y_t)-D^-c_j(x_i)] \geq 0 \quad .$$

The final inequality follows from the fact that for $0 \leq \delta \leq \epsilon$, $x(\delta) = x-\delta(u_j-u_i)$ is feasible for p and $g(x(\delta),r) \geq g(x,r)$. This establishes properties (i) and (ii). Property (iii) follows in the usual way.

Employing theorem 1, lemma 1, (18), and (19), we see that there is an x' with the property that

$$y_k-1 \leq y_k' \leq y_k$$

and

$$x_i-1 \leq x_i' \leq x_i \quad \text{for } i=1,2,\ldots,k \ ,$$

and

$$x_i' \geq x_i \quad \text{for } i=k+1,\ldots,n \quad .$$

But by the hypotheses of theorem 3, $g(x',r)$ is linear in x' for all x' satisfying the above constraints, i.e., there are numbers $a_o, a_1, \ldots, a_n$ such that

$$g(x',r) = a_o + \sum_{i=1}^{n} a_i x_i' \ .$$

Let S_1 be the set of indices i, $1 \leq i \leq k$, for which there is a feasible x' with $x_i' < x_i$. Let S_2 be the set of indices j, $k < j \leq n$, for which there is a feasible x' with $x_j' > x_j$. Since there is a feasible x' either S_1 and S_2 are empty, in which case $x' = x$ is feasible for p', or S_1 and S_2 are non empty.

Now if we fix y_k' at any feasible value between $y_k - 1$ and y_k, we may minimize $g(x',r)$ by finding an integer i in S_1 for which a_i is a maximum and an integer j in S_2 for which a_j is a minimum and letting $x' = x+(y_k-y_k')(u_j-u_i)$. Then $g(x',r) = g(x,r)+(y_k-y_k')(a_j-a_i)$. Thus to minimize $g(x',r)$ we let $y_k' = y_k$, i.e., $x' = x$, if $a_j \geq a_i$, and let $y_k' = y_k - 1$, i.e., $x' = x + u_j - u_i$, if $a_j < a_i$. This completes the proof.

An Algorithm for Automatic Computation

We now describe a procedure that starts with an x that is optimal for $p = (-r, \overline{x}, \underline{x}, \overline{y}, \underline{y})$ and that has integral coordinates. The algorithm then proceeds to find an x' that is optimal for $p' = (-r', \overline{x}', \underline{x}', \overline{y}', \underline{y}')$ and that has integral coordinates, or discovers that p' is not feasible. The algorithm involves three phases. In phase 1 the inventory and production constraints are relaxed, one at a time, until an optimal $\tilde{x}$ is found for the corresponding parameter set $\tilde{p} = (-r, \tilde{\bar{x}}, \tilde{\underline{x}}, \tilde{\bar{y}}, \tilde{\underline{y}})$ that has the property that $(\tilde{\bar{x}}, \tilde{\bar{y}}) \geq (\bar{x}', \bar{y}')$ and $(\tilde{\underline{x}}, \tilde{\underline{y}}) \leq (\underline{x}', \underline{y}')$. No infeasibilities can arise in phase 1. The second phase involves moving r "toward" r'. This is done so that at all times each intermediate requirements schedule $\hat{r}$, say, is such that $\tilde{\underline{X}}_n - \hat{R}_n \leq 0$. This phase terminates with an $\hat{x}$ that is optimal for $\hat{p} = (-r', \tilde{\bar{x}}, \tilde{\underline{x}}, \tilde{\bar{y}}, \tilde{\underline{y}})$, provided that $\hat{p}$ is feasible. A discussion of how infeasibilities are dealt with in phase 2 is deferred briefly. The third phase involves tightening the production and inventory constraints until p' is reached. If at any step in this final phase, an infeasible parameter set is found, then p' is not feasible.

We remark that with start 1, phase 2 (and often phase 3) is omitted. On the other hand with start 2, phase 1 (and often phase 3) is not needed.

An important simplification is possible in phases 1 and 3 whenever a production or inventory constraint is not binding. The idea is to take advantage of part (a) of lemma 1. For example if in phase 1 we find an $\tilde{x}$ that is optimal for $\tilde{p} = (-r, \tilde{\bar{x}}, \tilde{\underline{x}}, \tilde{\bar{y}}, \tilde{\underline{y}})$ and, say, $\tilde{y}_i < \tilde{\bar{y}}_i < \bar{y}_i'$, then we may immediately let $\tilde{\bar{y}}_i = \bar{y}_i'$ and still be assured that $\tilde{x}$ is optimal for the revised $\tilde{p}$. Similarly, if we find in phase 3 an $\hat{x}$ that is optimal for $\hat{p} = (-r', \hat{\bar{x}}, \hat{\underline{x}}, \hat{\bar{y}}, \hat{\underline{y}})$, and, say, $\hat{\underline{x}}_i < \hat{x}_i, \underline{x}_i'$, then we may immediately let $\underline{x}_i = \min(\hat{x}_i, \underline{x}_i')$ without disturbing the optimality of $\hat{x}$. These shortcuts avoid numerous applications of theorem 3 and are well worth using.

It remains to develop a means for dealing with infeasibilities in phase 2. In particular suppose that we seek an x that is optimal for $p = (-r, \bar{x}, \underline{x}, \bar{y}, \underline{y})$. We assume that p is feasible and has integral coordinates, and that P holds. Our approach to solving this problem, hereafter called problem A, is to construct a modified problem, called problem B, in which the parameter set is $\tilde{p} = (-r, \tilde{\bar{x}}, \underline{x}, \tilde{\bar{y}}, \tilde{\underline{y}})$ where $\tilde{\bar{x}}$, $\tilde{\bar{y}}$, and $\tilde{\underline{y}}$ are arbitrary but not effective bounds. In problem B the cost function is

$$\tilde{g}(x,r) = \sum_{i=1}^{n} \tilde{c}_i(x_i) + \sum_{i=1}^{n-1} \tilde{h}_i(y_i)$$

where

$$\tilde{c}_i(x_i) = \begin{cases} c_i(x_i) & , \ \underline{x}_i \le x_i \le \bar{x}_i \\ c_i(\bar{x}_i) + M(x_i - \bar{x}_i) , & \bar{x}_i < x_i \end{cases}$$

$$\tilde{h}_i(y_i) = \begin{cases} h_i(\underline{y}_i) - M(y_i - \underline{y}_i) , & y_i < \underline{y}_i \\ h_i(y_i) & , \ \underline{y}_i \le y_i \le \bar{y}_i \\ h_i(\bar{y}_i) + M(y_i - \bar{y}_i) , & \bar{y}_i < y_i \end{cases}$$

and M is a large positive constant. Notice that if x is feasible for p, then x is also feasible for $\tilde{p}$; furthermore, $\tilde{g}(x,r) = g(x,r)$.

Let

$$a = \max_{1 \le i \le n} D^- c_i(\bar{x}_i) ,$$

$$b^- = \max_{1 \le i < n} D^- h_i(\bar{y}_i), \quad b^+ = \max_{1 \le i < n} -D^+ h_i(\underline{y}_i) ,$$

$$\bar{c}_i = \max_{\underline{x}_i \le x_i \le \bar{x}_i} c_i(x_i), \quad \underline{c}_i = \min_{\underline{x}_i \le x_i \le \bar{x}_i} c_i(x_i) ,$$

$$\bar{h}_i = \max_{\underline{y}_i \le y_i \le \bar{y}_i} h_i(y_i), \quad \text{and} \quad \underline{h}_i = \min_{\underline{y}_i \le y_i \le \bar{y}_i} h_i(y_i) .$$

Let M be any number for which $(\bar{h}_n = \underline{h}_n = 0)$

$$M > \max \{a, b^-, b^+, \sum_{i=1}^{n} [\bar{c}_i + \bar{h}_i - \underline{c}_i - \underline{h}_i]\} \quad .$$

Since M is greater than $a, b^-, b^+,$ the $\tilde{c}_i$ and $\tilde{h}_i$ are convex functions. Hence, the modified cost functions satisfy P.

It follows from theorem 3 and the discussion thereafter that there exists an x that has integral coordinates and that is optimal for problem B. We now show that either x is feasible (and hence optimal) for problem A or that problem A has no feasible production schedule.

The proof is by contradiction. Suppose that x is not feasible for problem A, but that x^* (say) is feasible for that problem. Then for some i, $y_i \geq \bar{y}_i+1$, $y_i \leq \underline{y}_i-1$, or $x_i \geq \bar{x}_i+1$ since the y_i, $\underline{y}_i$, $\bar{y}_i$, x_i, and $\bar{x}_i$ are integers. Thus,

$$\begin{aligned} \tilde{g}(x,r) &\geq M + \sum_{i=1}^{n} (\underline{c}_i+\underline{h}_i) > \sum_{i=1}^{n} (\bar{c}_i+\bar{h}_i) \\ &\geq g(x^*,r) = \tilde{g}(x^*,r) \end{aligned}$$

which contradicts the optimality of x for problem B. The proof is complete.

In carrying out the computations, it is not necessary to know a suitable value of M. Instead, when a collection of new schedules is examined, as in applying theorem 3, one proceeds as follows. First locate the "least infeasible" schedules in the class being examined, i.e., the schedules for which the sum of the amounts by which the x_i and y_i respectively exceed the $\bar{x}_i$ and $\bar{y}_i$, and the amount by which the y_i fall below the $\underline{y}_i$, is minimal. Then, letting $M = 0$, choose a minimal cost schedule from among those located using $\tilde{g}$ as the cost function.

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